# Calculating the radius of rolled tape

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J.L.

Once upon a time it happened that I got delivered C60 cassettes, but some of them seemed to have less tape than others. The playing time for all cassettes was the full 60 minutes, but there was noticeable difference in the amount of packed tape in the tape reel. The assumption was that thinner C90 tape was accidentally rolled inside the C60 cassette. Then this little mathematical puzzle needed to be solved.

Here, a generic equation is derived for calculating the total radius of the outermost circle, when lenght $L$ of tape, sheet of paper or any flat string like object having thickness of $h$ is rolled around a cylinder of radius $r_0$. The resulting roll is assumed to be packed tight, without gaps between the rolled layers.

The ’problem’ should be made more clear using an illustration, which is given in Figure 1. The velocity $v$ is just a helping parameter when setting up a

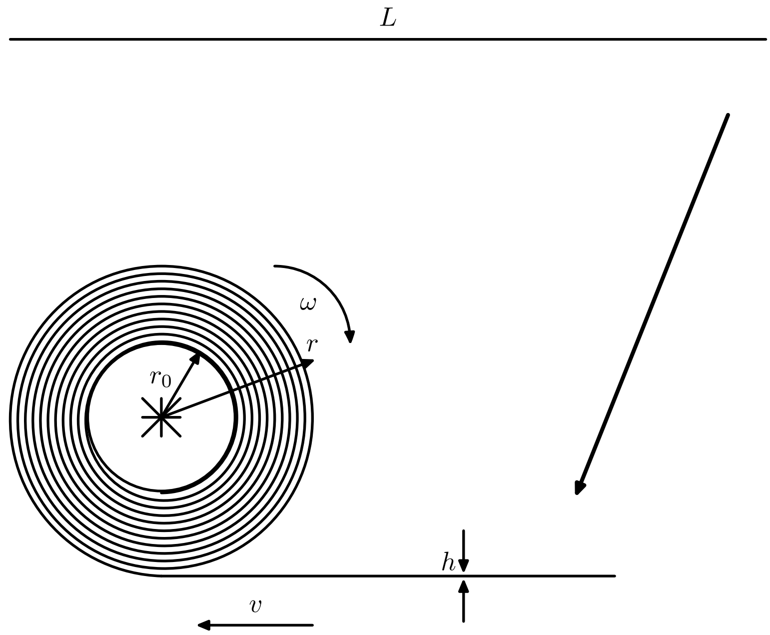


Figure 1: Tape of lenght $L$ is rolled around a cylinder with radius $r_0$

reasonable differential equation and is assumed to be an arbitrary constant value.

Differential equation describing the rate of change of the radius in the rolling process is:

$$\frac{dr}{dt} = \frac{v}{2\pi r} h, \tag{1}$$

where with every full revolution $2\pi r$, the radius grows by thickness $h$. Note that the change of radius depends on itself.

Separating the variables results to a basic first-order differential equation

$$r\,dr = \frac{v}{2\pi} h\,dt. \tag{2}$$

Integrating both sides yields

$$\int r\,dr = \frac{vh}{2\pi}\int dt \qquad \Longrightarrow \qquad r^2 = \frac{vh}{\pi}(t + C), \tag{3}$$

where $C$ is the constant of integration.

It is clear that at time $t = 0$, the radius is $r_0$. Using that information, the value of $C$ can be evaluated:

$$r_0^2 = \frac{vh}{\pi} C \qquad \Longrightarrow \qquad C = \frac{\pi}{vh} r_0^2. \tag{4}$$

Then the complete solution is

$$r^2 = \frac{vh}{\pi} t + r_0^2 \qquad \Longrightarrow \qquad r(t) = \sqrt{r_0^2 + \frac{vh}{\pi} t}. \tag{5}$$

Then we notice that velocity and time are totally useless parameters in this equation. To fix that, one can replace the constant velocity with distance per time unit, namely

$$r = \sqrt{r_0^2 + \frac{x_1 - x_0}{t}\frac{h}{\pi} t}, \tag{6}$$

and when simplified, equates to:

$$\boxed{r = \sqrt{r_0^2 + \frac{Lh}{\pi}}} \tag{7}$$

This equation does not feel intuitive, because it is not clearly seen how it contains the logic of spinning the tape around itself so that the full lenght of tape is rolled up and at each round of rotation the radius grows by the thickness. Basically all the parameters are in the equation, but how does this actually work... It seems that $\pi$ does some magic here that is not easily understandable.